

An estimation procedure for the Linnik distribution

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Abstract We propose estimators for the parameters of the Linnik $L(\alpha, \gamma)$ distribution. The estimators are derived from the moments of the log-transformed Linnik distributed random variable, and are shown to be asymptotically unbiased. The estimation algorithm is computationally simple and less restrictive. Our procedure is also tested using simulated data.

Keywords Linnik · geometric stable · estimation · financial modeling · economics

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1 Introduction

In recent years, the Linnik $L(\alpha, \gamma)$ distribution of [18] has gained popularity from researchers in many scientific areas. For instance, it has been used to model random phenomena in finance (e.g., S&P index) [12,13]. In addition, [6,13,14,17,19] (and the references therein) studied the $L(\alpha, \gamma)$ probability density function with characteristic function (ch.f.)

$$\psi(\lambda) = (1 + |\gamma\lambda|^\alpha)^{-1}, \quad (1.1)$$

where $\gamma > 0$ is the scale parameter, $\lambda \in \mathbb{R}$, and $0 < \alpha \leq 2$. In particular, the probability density and the cumulative distribution functions for a $L(\alpha, 1)$

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distributed random variable are

$$f(x) = \frac{\sin(\pi\alpha/2)}{\pi} \int_0^\infty \frac{y^\alpha \exp(-xy)}{y^{2\alpha} + 2y^\alpha \cos(\alpha\pi/2) + 1} dy,$$

and

$$F(x) = 1 - \frac{\sin(\pi\alpha/2)}{\pi} \int_0^\infty \frac{y^{\alpha-1} \exp(-xy)}{y^{2\alpha} + 2y^\alpha \cos(\alpha\pi/2) + 1} dy, \quad x > 0,$$

correspondingly. Moreover, [13] constructed the following structural representation of a $L(\alpha, \gamma)$ distributed random variable L :

$$L \stackrel{d}{=} \gamma D R^{1/\alpha}, \quad (1.2)$$

where D has the standard Laplace distribution (with location 0 and scale 1), and R has the density function

$$f_R(r) = \frac{\sin(\pi\rho)}{\rho\pi [r^2 + 2r \cos(\rho\pi) + 1]}, \quad 0 < \rho < 1, \quad r > 0.$$

It is known that the $L(\alpha, \gamma)$ distribution is geometric stable.

The parameter estimation problem for α when $\gamma = 1$ was addressed by [1] using the methods of [16, 20, 21]. Then [8] adopted [21]'s technique to estimate the parameters α and γ of the Linnik $L(\alpha, \gamma)$ distribution. Furthermore, [8] constructed estimators that require choosing values of λ , λ_i 's say such that $\lambda_i \in \mathbb{R} \setminus 0$, $i = 1, \dots, b$, and $\sum_{j=1}^b (\log |\lambda_j| - c)^2 > 0$, where $c = (1/b) \sum_{j=1}^b \log |\lambda_j|$. More importantly, [8] deduced that λ_i 's should be restricted to a region where $\log(|\hat{\psi}(\lambda)| - 1)$ is 'linear' with respect to $\log |\lambda|$ to obtain satisfactory results. Apparently, satisfying the above restrictions is not straightforward in practice. Note that $\psi(\lambda)$ is the method-of-moments estimator of the characteristic function (1.1). Moreover, they showed the asymptotic normality of their point estimators, i.e.,

$$\sqrt{n} \begin{pmatrix} \hat{\alpha}_P - \alpha \\ \hat{\gamma}_P - \gamma \end{pmatrix} \xrightarrow{d} N[\mathbf{0}, \mathbf{W}_P]$$

as $n \rightarrow \infty$, where the entries of the covariance matrix $\mathbf{W}_P = (\mathbf{W}_{P_{ij}})$ are defined as

$$\mathbf{W}_{P_{11}} = \mathbf{u}^T \mathbf{W} \mathbf{u}, \quad \mathbf{W}_{P_{12}} = \mathbf{W}_{P_{21}} = \frac{\gamma}{\alpha} (\mathbf{1}/b - \mathbf{u}(c + \log \gamma))^T \mathbf{W} \mathbf{u},$$

$$\mathbf{W}_{P_{22}} = \left(\frac{\gamma}{\alpha}\right)^2 (\mathbf{1}/b - \mathbf{u}(c + \log \gamma))^T \mathbf{W} (\mathbf{1}/b - \mathbf{u}(c + \log \gamma)),$$

$\mathbf{W} = (w_i w_j w_{ij})$, $w_i = (\psi(\lambda_i)^{-2}) / |\gamma \lambda_i|^\alpha$, $\mathbf{1} = (1, \dots, 1)^T$, $u_i = (\log |\lambda_i| - c) / \sum_{j=1}^b (\log |\lambda_j| - c)^2$, $\mathbf{u}^T = (u_1, \dots, u_b)$, and $w_{ij} = [\psi(\lambda_i + \lambda_j) + \psi(\lambda_i - \lambda_j) - 2\psi(\lambda_i)\psi(\lambda_j)]/2$. The point estimators of α and λ are given by

$$\hat{\alpha}_P = \sum_{j=1}^b \log(|\hat{\psi}|^{-1} - 1) u_j$$

and

$$\hat{\gamma}_P = \exp \left\{ \frac{1}{b\hat{\alpha}_P} \sum_{j=1}^b \log \left(|\hat{\psi}|^{-1} - 1 \right) - c \right\},$$

correspondingly. Observe that one can always use the tail-index estimators of [9,10,11] for the parameter α as well. However, these methods impose restrictions or use a portion of the data only making them less efficient.

Similarly, [13] suggested the fractional moment estimators, i.e., choose constants $q_1, q_2 < \alpha$, and calculate the parameter estimates $\hat{\alpha}_K$ and $\hat{\gamma}_K$ by solving the following system of two non-linear equations:

$$\hat{\mu}_{|L|^{q_j}} = \widehat{\mathbf{E}|L|^{q_j}} = \frac{\pi q_j (1 - q_j) \hat{\gamma}_K^{q_j}}{\hat{\alpha}_K \Gamma(2 - q_j) \sin(\pi q_j / \hat{\alpha}_K) \cos(\pi q_j / 2)}, \quad j = 1, 2.$$

Clearly, the pre-selection of appropriate values requires information about the true or unknown parameter α a priori, which is not feasible in practice. As a direct consequence, it is expected that the above estimators will perform poorly when the restrictions are violated. Notice also that the point estimators of [8] have complicated expressions while [13]'s is computationally involved. It is mainly these drawbacks that stimulate us to construct a simple estimation algorithm that uses all the available information possible, and avoids the above limitations.

The main goal of this paper is to propose estimators for the parameters α and γ of the Linnik $L(\alpha, \gamma)$ distribution. The rest of the paper is organized as follows: In Section 2, we review a structural representation of a Linnik $L(\alpha, \gamma)$ random variable via symmetric stable. In Section 3, we derive the method-of-moments estimators. In Section 4, the asymptotic normality of the estimators are then shown. Empirical test results are given in Section 5. Section 6 discusses key points and extensions of our study.

2 Stable representation of the random variable L

For the sake of completeness, we review the stable representation of the random variable L in (1.1) as derived by [6].

Result 1 *Let $0 < \alpha \leq 2$, and the scale parameter $\gamma > 0$. Then*

$$L \stackrel{d}{=} \gamma Z^{1/\alpha} S, \quad 0 < q < \alpha \quad (2.1)$$

where S is a symmetric stable distributed random variable with the characteristic function $\psi_S(\lambda) = \exp(-|\lambda|^\alpha)$, and Z is an exponentially (with scale 1) and independently distributed random variable.

An expression for the q th fractional moment can be straightforwardly derived from the above result, and is given below.

Remark 1 Let $0 < \alpha \leq 2$, and the scale parameter $\gamma > 0$. Then

$$\mathbf{E}|L|^q = \frac{\pi q \gamma^q}{\alpha \sin(\pi q/\alpha) \cos(\pi q/2) \Gamma(1-q)}.$$

Proof. The q th fractional moment is

$$\mathbf{E}|L|^q = \gamma^q \mathbf{E}|S|^q \mathbf{E}(Z^{q/\alpha}).$$

Using the q th fractional moment of the symmetric stable random variable S (see [4])

$$w_\alpha(q) = \mathbf{E}|S|^q = \frac{\Gamma(1-q/\alpha)}{\cos(q\pi/2) \Gamma(1-q)}, \quad (2.2)$$

we have

$$\mathbf{E}|L|^q = \gamma^q \left(\frac{\Gamma(1-q/\alpha)}{\cos(q\pi/2) \Gamma(1-q)} \right) \Gamma(1+q/\alpha).$$

Substituting

$$\Gamma(1-p/\alpha) = \frac{\pi}{\Gamma(q/\alpha) \sin(\pi q/\alpha)}, \quad \text{and} \quad \Gamma(1+q/\alpha) = \frac{q}{\alpha} \Gamma(q/\alpha)$$

into the preceding equation completes the proof. Observe that the q th fractional moment above has a different and simpler form in comparison with [13].

3 Method-of-Moments (MoM) estimation

Applying the log transformation to the absolute value of the random variable L given in (2.1), we get

$$L' \stackrel{d}{=} \log(\gamma) + \frac{1}{\alpha} Z' + S', \quad (3.1)$$

where $L' = \log(|L|)$, $Z' = \log(Z)$, and $S' = \log(|S|)$. From [5], we have

$$\mathbf{E}(Z') = -\mathbb{C}, \quad \mathbf{E}(Z')^2 = \mathbb{C}^2 + \frac{\pi^2}{6},$$

$$\mathbf{E}(Z')^3 = -\mathbb{C}^3 - \frac{\mathbb{C}\pi^2}{2} - 2\zeta(3), \quad \mathbf{E}(Z')^4 = \mathbb{C}^2(\mathbb{C}^2 + \pi^2) + \frac{3\pi^4}{20} + 8\mathbb{C}\zeta(3),$$

where $\mathbb{C} \simeq 0.5772156649015328606065$ is the Euler's constant. Recall the following formula from [4] and [22] for the higher log-moments of $|S|$:

$$\mathbf{E}(S')^k = (d^k w_\alpha(q)/dq^k) \big|_{q=0},$$

where $w_\alpha(q)$ is defined in (2.2). We now need to find the power series expansion of $w_\alpha(q)$. But this turns out to be easier if we first expand

$$\log w_\alpha(q) = \log \Gamma(1 - q/\alpha) - \log \Gamma(1 - q) - \log \cos(q\pi/2)$$

into a power series. Using the well-known expansions

$$\log \Gamma(1 - \theta) = \mathbb{C}\theta + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} \theta^k,$$

and

$$\log \cos(\theta) = -\theta^2/2 + O(\theta^4),$$

we get

$$\log w_\alpha(q) = \mathbb{C} \left(\frac{1}{\alpha} - 1 \right) q + \frac{\pi^2 (\alpha^2 + 2)}{24\alpha^2} q^2 + \frac{(1 - \alpha^3) \zeta(3)}{3\alpha^3} q^3 + \frac{(8 + 7\alpha^4) \pi^4}{2880\alpha^4} q^4 + O(q^5),$$

where $\zeta(\theta)$ is the Riemann zeta function evaluated at θ . This implies that

$$\begin{aligned} w_\alpha(q) &= 1 + \mathbb{C} \left(\frac{1}{\alpha} - 1 \right) q + \left[\frac{12\mathbb{C}^2(\alpha - 1)^2 + (\alpha^2 + 2) \pi^2}{24\alpha^2} \right] q^2 \\ &\quad + \left[\frac{(1 - \alpha) (4(\alpha - 1)^2 \mathbb{C}^3 + (\alpha^2 + 2) \mathbb{C} \pi^2 + 8(\alpha^2 + \alpha + 1) \zeta(3))}{24\alpha^3} \right] q^3 \\ &\quad + \frac{1}{5760\alpha^4} \left[240(\alpha - 1)^4 \mathbb{C}^4 + 120(\alpha - 1)^2 (\alpha^2 + 2) \mathbb{C}^2 \pi^2 \right. \\ &\quad \left. + (19\alpha^4 + 20\alpha^2 + 36) \pi^4 + 1920(\alpha - 1)^2 (\alpha^2 + \alpha + 1) \mathbb{C} \zeta(3) \right] q^4 + O(q^5). \end{aligned}$$

The k th log-moment is simply the coefficient of the term $q^k/k!$ in the above power series expansion. In particular, the first four integer-order log-moments can be easily deduced as

$$\begin{aligned} \mathbf{E} \left(S'_\alpha \right) &= \mathbb{C} \left(\frac{1}{\alpha} - 1 \right), \quad \mathbf{E} \left(S'_\alpha \right)^2 = \frac{12\mathbb{C}^2(\alpha - 1)^2 + (\alpha^2 + 2) \pi^2}{12\alpha^2}, \\ \mathbf{E} \left(S'_\alpha \right)^3 &= \frac{(1 - \alpha) (4(\alpha - 1)^2 \mathbb{C}^3 + (\alpha^2 + 2) \mathbb{C} \pi^2 + 8(\alpha^2 + \alpha + 1) \zeta(3))}{4\alpha^3}, \quad \text{and} \\ \mathbf{E} \left(S'_\alpha \right)^4 &= \frac{1}{240\alpha^4} \left[240(\alpha - 1)^4 \mathbb{C}^4 + 120(\alpha - 1)^2 (\alpha^2 + 2) \mathbb{C}^2 \pi^2 \right. \\ &\quad \left. + (19\alpha^4 + 20\alpha^2 + 36) \pi^4 + 1920(\alpha - 1)^2 (\alpha^2 + \alpha + 1) \mathbb{C} \zeta(3) \right]. \end{aligned}$$

Using the above moments and the structural equality (3.1), we get the mean and variance

$$\mu_{L'} = \log(\gamma) - \mathbb{C}, \quad \text{and} \quad \sigma_{L'}^2 = \frac{\pi^2(\alpha^2 + 4)}{12\alpha^2}, \quad (3.2)$$

respectively. Hence, we have the MoM estimators of α and γ as

$$\hat{\alpha} = \frac{\pi}{\sqrt{3 \left(\hat{\sigma}_{L'}^2 - \pi^2/12 \right)}}, \quad \text{and} \quad \hat{\gamma} = \exp(\hat{\mu}_{L'} + \mathbb{C}), \quad (3.3)$$

correspondingly. Moreover, a similar calculation gives the third and fourth central moments of S' as

$$\mu_3' = \mathbf{E} \left(L' - \mu_{L'} \right)^3 = -2\zeta(3),$$

and

$$\mu_4' = \mathbf{E} \left(L' - \mu_{L'} \right)^4 = \frac{\pi^4(19\alpha^4 + 40\alpha^2 + 112)}{240\alpha^4},$$

which are useful in the derivation of the interval estimates given in the next section.

4 Asymptotic normality of the estimators $\hat{\alpha}$ and $\hat{\gamma}$

We will now show that the estimators (3.3) of α and γ are asymptotically normal. Let

$$\hat{\mu}_{L'} = \overline{L'} = \frac{\sum_{j=1}^n L_j'}{n} \quad \text{and} \quad \hat{\sigma}_{L'}^2 = \frac{\sum_{j=1}^n \left(L_j' - \overline{L'} \right)^2}{n}.$$

Then the following weak convergence holds (see [7]), i.e.,

$$\sqrt{n} \begin{pmatrix} \hat{\mu}_{L'} - \mu_{L'} \\ \hat{\sigma}_{L'}^2 - \sigma_{L'}^2 \end{pmatrix} \xrightarrow{d} N[\mathbf{0}, \mathbf{\Sigma}]$$

as $n \rightarrow \infty$, where the covariance matrix $\mathbf{\Sigma}$ is defined as

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_{L'}^2 & \mu_3' \\ \mu_3 & \mu_4' - \sigma_{L'}^4 \end{pmatrix},$$

μ_3' , μ_4' , and $\sigma_{L'}^2$ are given in Section 3. Using a standard result on asymptotic theory, the two-dimensional Central Limit Theorem above implies that

$$\sqrt{n}(\mathbf{g}(\hat{\boldsymbol{\theta}}_n) - \mathbf{g}(\boldsymbol{\theta})) \xrightarrow{d} N(0, \dot{\mathbf{g}}(\boldsymbol{\theta})^T \mathbf{\Sigma} \dot{\mathbf{g}}(\boldsymbol{\theta})),$$

where $\hat{\boldsymbol{\theta}}_n = (\hat{\mu}_{L'}, \hat{\sigma}_{L'}^2)^T$, \mathbf{g} is a mapping from $\mathbb{R}^2 \rightarrow \mathbb{R}$, and $\dot{\mathbf{g}}(\mathbf{x})$ is continuous in a neighborhood of $\boldsymbol{\theta} \in \mathbb{R}^2$. We now apply this result to the consistent estimator of γ . Letting

$$\mathbf{g}(\mu_{L'}, \sigma_{L'}^2) = \exp(\mu_{L'} + \mathbb{C}).$$

Then the gradient becomes

$$\dot{\mathbf{g}}(\mu_{L'}, \sigma_{L'}^2) = \begin{pmatrix} \exp(\mu_{L'} + \mathbb{C}) \\ 0 \end{pmatrix}.$$

This implies that

$$\sqrt{n}(\hat{\gamma} - \gamma) \xrightarrow{d} N[0, \sigma_\gamma^2],$$

where

$$\begin{aligned} \sigma_\gamma^2 &= \dot{\mathbf{g}}(\mu_{L'}, \sigma_{L'}^2)^\top \begin{pmatrix} \sigma_{L'}^2 & \mu_3' \\ \mu_3 & \mu_4' - \sigma_{L'}^4 \end{pmatrix} \dot{\mathbf{g}}(\mu_{L'}, \sigma_{L'}^2) \\ &= \frac{\pi^2 e^{2(\mu_{L'} + \mathbb{C})} (\alpha^2 + 4)}{12\alpha^2} \\ &= \frac{\pi^2 \gamma^2 (\alpha^2 + 4)}{12\alpha^2}, \end{aligned}$$

and the last line is obtained by substituting $(\log(\gamma) - \mathbb{C})$ for $\mu_{L'}$. Similarly,

$$\begin{aligned} \sqrt{n}(\hat{\alpha} - \alpha) &\xrightarrow{d} N \left[0, \left(\frac{\pi}{2\sqrt{3}(\sigma_{L'}^2 - \pi^2/12)^{3/2}} \right)^2 (\mu_4' - \sigma_{L'}^4) \right] \\ &= N \left[0, \frac{\alpha^2}{80} (13\alpha^4 + 20\alpha^2 + 64) \right], \end{aligned}$$

where the final simplification is attained by plugging in $\sigma_{L'}^2 = \frac{\pi^2(\alpha^2+4)}{12\alpha^2}$ (see 3.2),

$$\mathbf{g}(\mu_{L'}, \sigma_{L'}^2) = \frac{\pi}{\sqrt{3(\sigma_{L'}^2 - \pi^2/12)}},$$

and

$$\dot{\mathbf{g}}(\mu_{L'}, \sigma_{L'}^2) = \begin{pmatrix} 0 \\ \pi / \left(2\sqrt{3}(\sigma_{L'}^2 - \pi^2/12)^{3/2} \right) \end{pmatrix}.$$

Therefore, we have shown that our method-of-moments estimators are normally distributed (asymptotically unbiased) as the sample size n goes large. Consequently, we can now approximate the $(1 - \varepsilon)100\%$ confidence interval for α and γ as

$$\hat{\alpha} \pm z_{\varepsilon/2} \sqrt{\frac{\hat{\alpha}^2 (13\hat{\alpha}^4 + 20\hat{\alpha}^2 + 64)}{80n}},$$

and

$$\hat{\gamma} \pm z_{\varepsilon/2} \sqrt{\frac{\pi^2 \hat{\gamma}^2 (\hat{\alpha}^2 + 4)}{12\hat{\alpha}^2 n}},$$

respectively, where $z_{\varepsilon/2}$ is the $(1 - \varepsilon/2)$ th quantile of the standard normal distribution, and $0 < \varepsilon < 1$.

5 Testing MoM estimators on simulated data

In this section, we computationally test the MoM estimators of α and γ obtained in Section 4 using the median absolute deviation (MAD) from the true values of our parameters as our criterion. We generated 2000 samples of sizes $n = 100, 1000$, and 10000 . The estimates $\hat{\alpha}$ and $\hat{\gamma}$ for each of the m samples and the average are then calculated. These values are shown in Table 1 below. When the sample size is at least $n = 10000$, the relative fluctuations of the estimates of α and γ are becoming less than 2.2% and 17.7%, respectively. Notice that the estimator of γ has a large variation when α is close to 0. It is also seemingly biased for both small values of α and small sample sizes. Nevertheless, it can be seen from Table 1 that the point estimators are asymptotically unbiased as expected.

Table 1 Mean estimates of and dispersions from the true parameters α and γ .

(α, γ)		$n = 100$		$n = 1000$		$n = 10000$	
		Mean	MAD	Mean	MAD	Mean	MAD
(0.1, 0.05)	α	0.101	0.009	0.100	0.003	0.100	0.001
	γ	0.250	0.071	0.058	0.027	0.051	0.009
(0.2, 0.5)	α	0.202	0.017	0.200	0.006	0.200	0.002
	γ	0.736	0.400	0.518	0.141	0.502	0.048
(0.5, 1000)	α	0.507	0.047	0.501	0.014	0.500	0.004
	γ	1056.029	344.901	1007.287	116.633	1000.422	35.313
(0.8, 100)	α	0.811	0.082	0.801	0.026	0.800	0.008
	γ	103.865	24.042	100.382	7.803	100.021	2.455
(1, 0.2)	α	1.018	0.107	1.001	0.035	1.000	0.010
	γ	0.203	0.039	0.200	0.012	0.200	0.004
(1.2, 10)	α	1.234	0.146	1.202	0.045	1.200	0.014
	γ	10.162	1.757	10.023	0.581	9.997	0.179
(1.75, 1)	α	1.845	0.317	1.760	0.096	1.750	0.031
	γ	1.011	0.142	1.000	0.041	1.000	0.013
(2, 0.1)	α	2.170	0.429	2.014	0.130	2.000	0.042
	γ	0.101	0.012	0.100	0.004	0.100	0.001

In the interval calculations, we simulated 2000 sets of sample size n and averaged the lower and upper 95% confidence bounds using the formula obtained in Section 4. Table 2 below shows the asymptotic behavior of the confidence intervals using sample sizes $n = 100, 1000$, and 10000 . Generally, it can be seen that the asymptotic interval estimators performed quite satisfactorily for different combinations of the parameter values. We emphasize that one can always use bootstrap methods as the explicit forms of the estimators are known. But the asymptotic-based interval estimates are faster and easier to compute.

Table 3 shows the corresponding coverage probabilities of the interval estimates above. Apparently, the proposed interval estimator of α performed relatively well near the boundaries for sample size $n = 100$. Note that [8]’s interval estimator performs satisfactorily only when $\alpha < 1$ even with sample size $n = 500$. In addition, the table below insinuates that if the true parameter

Table 2 Average 95% confidence intervals for different values of α and γ .

		$n = 100$	$n = 1000$	$n = 10000$
(0.1, 0.05)	α	(0.083 , 0.118)	(0.095 , 0.105)	(0.098 , 0.101)
	γ	(-0.733 , 1.297)	(-0.008 , 0.127)	(0.033 , 0.068)
(0.2, 0.5)	α	(0.167 , 0.239)	(0.189 , 0.211)	(0.196 , 0.203)
	γ	(-0.588 , 2.086)	(0.213 , 0.830)	(0.412 , 0.591)
(0.5, 1000)	α	(0.411 , 0.596)	(0.471 , 0.529)	(0.491 , 0.509)
	γ	(290.060 , 1844.721)	(772.112 , 1237.549)	(926.761 , 1073.340)
(0.8, 100)	α	(0.649 , 0.974)	(0.750 , 0.851)	(0.785 , 0.816)
	γ	(54.162 , 152.626)	(85.195 , 115.581)	(95.334 , 104.913)
(1, 0.2)	α	(0.796 , 1.247)	(0.932 , 1.068)	(0.978 , 1.021)
	γ	(0.124 , 0.285)	(0.175 , 0.225)	(0.192 , 0.208)
(1.2, 10)	α	(0.920 , 1.529)	(1.111 , 1.293)	(1.171 , 1.228)
	γ	(6.644 , 13.594)	(8.916 , 11.101)	(9.654 , 10.344)
(1.75, 1)	α	(1.083 , 4.003)	(1.563 , 1.949)	(1.691 , 1.811)
	γ	(0.733 , 1.268)	(0.915 , 1.085)	(0.973 , 1.026)
(2, 0.1)	α	(0.925 , 3.448)	(1.748 , 2.282)	(1.920 , 2.085)
	γ	(0.076 , 0.126)	(0.092 , 0.108)	(0.098 , 0.102)

value of α is close to 0 then we need at least 1000 observations to obtain reasonable estimates for small values of γ . Nonetheless, the coverage probabilities still provide relatively good merits for our estimators especially for sample size $n = 100$.

Table 3 Coverage probabilities of 95% interval estimates for different values of α and γ .

		$n = 100$	$n = 1000$	$n = 10000$
(0.1, 0.05)	α	0.954	0.948	0.949
	γ	0.803	0.914	0.947
(0.2, 0.5)	α	0.945	0.943	0.954
	γ	0.859	0.944	0.955
(0.5, 1000)	α	0.959	0.952	0.948
	γ	0.933	0.945	0.946
(0.8, 100)	α	0.961	0.956	0.951
	γ	0.944	0.945	0.946
(1, 0.2)	α	0.953	0.949	0.955
	γ	0.949	0.938	0.948
(1.2, 10)	α	0.958	0.942	0.943
	γ	0.943	0.949	0.946
(1.75, 1)	α	0.942	0.956	0.955
	γ	0.951	0.946	0.953
(2, 0.1)	α	0.940	0.959	0.947
	γ	0.946	0.949	0.943
Average	α	0.951	0.950	0.950
	γ	0.916	0.941	0.948

We now attempt to compare our procedure with that of [8] and [13] by using the mean point estimate (unbiasedness for finite samples) and coefficient of variation (CV) as criteria. Additionally, we followed [13] by focusing on $\alpha \in (1, 2]$ for simplicity and due to the likely applications in finance

in this range. We utilized the same constants $q_1 = 0.5, q_2 = 1$ of [13], and $\lambda_1 = 0.001, \lambda_2 = 0.1$ of [8]. The results are showcased in Table 4 below using 2000 samples of size $n = 100, 1000, 10000$ each. It can be seen that the maximum CV of the proposed estimators is 26.85% while [8] and [13] have 44.05% and 40.11%, correspondingly. Undoubtedly, the proposed procedure outperformed the competing methods in estimating γ . It also performed best in estimating α except when the true value is close to 2 and the sample size $n < 10000$. Likewise, we generated 10000 samples of size $n = 100$, and obtained the coverage probabilities 95.1% and 91.7% for $\alpha = 0.4$ and $\gamma = 0.5$, respectively compared with the 93.1% and 86.5% of [8]. These consequences considerably provide an additional argument in favor of the proposed approach. On the contrary, it appears that [8]'s procedure needs further tuning of the λ values to obtain better results.

Table 4 Comparison of point estimators for different values of α, γ , and sample size n .

(α, γ)	n		$\hat{\alpha}$	$\hat{\alpha}_K$	$\hat{\alpha}_P$	$\hat{\gamma}$	$\hat{\gamma}_K$	$\hat{\gamma}_P$
(1.1, 0.9)	100	Mean	1.125	1.328	1.632	0.919	1.142	1.927
		CV(%)	11.82	10.18	22.91	18.81	40.11	40.18
	1000	Mean	1.104	1.228	1.290	0.900	1.037	1.285
		CV(%)	3.64	6.34	22.33	5.89	8.74	44.05
	10000	Mean	1.100	1.180	1.126	0.900	0.993	0.960
		CV(%)	1.16	4.37	9.57	1.87	9.57	22.63
(1.3, 2)	100	Mean	1.341	1.468	1.699	2.038	2.223	2.861
		CV(%)	13.99	10.78	18.92	16.94	17.13	29.61
	1000	Mean	1.301	1.374	1.468	2.002	2.103	2.382
		CV(%)	4.14	6.62	16.57	5.16	6.14	24.99
	10000	Mean	1.300	1.338	1.334	2.000	2.054	2.077
		CV(%)	1.29	4.39	8.33	1.62	4.22	13.15
(1.5, 1)	100	Mean	1.555	1.617	1.896	1.013	1.049	1.574
		CV(%)	16.00	10.12	11.17	15.45	14.38	27.23
	1000	Mean	1.504	1.547	1.759	1.001	1.018	1.401
		CV(%)	4.72	5.97	12.64	4.81	5.31	24.01
	10000	Mean(%)	1.500	1.514	1.603	1.000	1.006	1.164
		CV(%)	1.51	3.74	11.6	1.51	3.07	24.14
(1.7, 10)	100	Mean	1.791	1.779	1.836	10.127	10.240	10.077
		CV(%)	20.36	8.43	8.85	14.61	13.04	13.91
	1000	Mean	1.707	1.721	1.752	10.005	10.045	9.997
		CV(%)	5.52	5.09	7.30	4.52	4.59	4.51
	10000	Mean	1.701	1.705	1.710	9.995	10.005	9.995
		CV(%)	1.70	2.79	3.70	1.38	2.04	1.41
(1.99, 0.1)	100	Mean	2.129	2.010	2.000	0.101	0.101	0.100
		CV(%)	26.85	5.52	0.00	12.56	11.11	11.89
	1000	Mean	1.998	1.990	1.999	0.100	0.100	0.102
		CV(%)	6.78	2.01	0.13	4.15	3.64	12.05
	10000	Mean	1.991	1.990	1.999	0.100	0.100	0.101
		CV(%)	2.06	0.67	0.93	1.29	1.15	3.82

Overall, Tables 1-4 strongly indicate that the proposed point and interval estimators did fairly well in our simulations. The result of the comparison

further added merits to our method. The point estimates could be regarded as reasonable starting values for better iterative estimation algorithms too.

6 Concluding remarks

We have derived the first few moments of a log-transformed Linnik distributed random variable. The derivation led to a system of estimating equations which are then used to estimate the parameters of the $L(\alpha, \gamma)$ distribution. A major advantage of our estimators is that we do not need to choose appropriate constants a priori to obtain reasonable parameter estimates. We emphasized that the pre-selection of these constants is not straightforward in practice. Furthermore, the proposed estimators are computationally simpler and faster as we do not need to solve a system of non-linear equations using iterative numerical methods. In general, the proposed estimators outperformed the competing procedures.

To conclude, we cite a few extensions which would be worth pursuing in the future. For instance, improving the above estimators by bootstrapping, and developing other estimators using the likelihood approach would be of interest. The derivation of the corresponding method-of-moments estimators for the multivariate case as in [1, 3, 15], and the application of our procedure in practice particularly in finance and economics would be valuable pursuits as well.

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